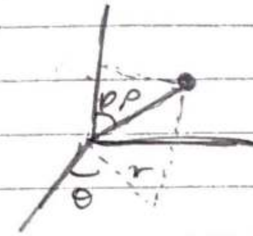
II. Spherical Coordinates

IDEA: Every point $\in \mathbb{R}^3$ lives on a sphere



We parametrize via:

ρ = distance from (x,y,z) to origin

θ = angle from pos x-axis to point $(x,y,0)$

φ = angle from pos z-axis to point (x,y,z)

$$\boxed{x = r\cos\theta = \rho\sin(\varphi)\cos\theta \qquad z = \rho\cos(\varphi) \qquad y = r\sin\theta = \rho\sin(\varphi)\sin(\theta)}$$

Check: $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \boxed{\rho^2\sin(\varphi)}$

--- Guest lecture ---

Jacobian for Spherical Coordinates:
using x, y, z as above

$$J = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$=1$

$$= \cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta\right)$$

$$+ \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$=1$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^2\phi\sin\phi = \boxed{\rho^2\sin\phi}$$

$x^2+y^2+z^2=\rho^2$

Ex) Compute $\iiint_R (x^2+y^2+z^2)^2\, dV$ where R is the solid ball of radius 5 about the origin

$R_{sph} = \{(\rho,\phi,\theta) : 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

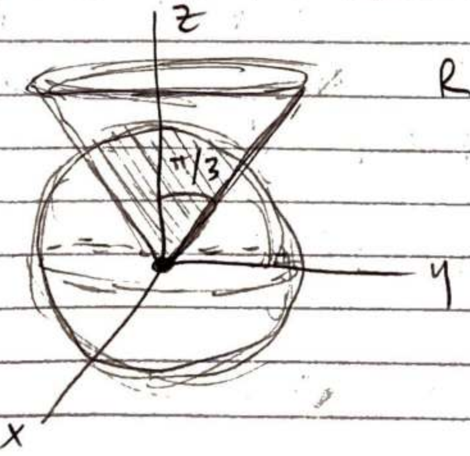
$$\int_{\rho=0}^{5}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} (\rho^2)^2 \cdot \rho^2 \sin\phi \, dA\ d\phi\ d\theta\ d\rho$$

Inner: $\int_{\phi=0}^{\pi} \rho^6 \sin\phi = -\rho^6 \cos\phi \Big|_0^{\pi} = 2\rho^6 - 0$

Middle: $\int_{\theta=0}^{2\pi} 2\rho^6 = 2\rho^6\theta \Big|_{\theta=0}^{2\pi} = 4\pi\rho^6 - 0$

Outer: $\int_{\rho=0}^{5} 4\pi\rho^6 = 4\pi \frac{\rho^7}{7} = \boxed{4\pi \frac{5^7}{7}} - 0$

Ex: $\iiint_R (y^2 z)\, dV$, R is the region above the cone w/ point at the origin and making an angle of $\pi/3$ rad w/ the positive z-axis, AND inside sphere w/ radius 2 centered at the origin



$R: (\rho,\phi,\theta) : 0 \le \rho \le 2,$
$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3$

$$\int_{\rho=0}^{2}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{3}} (\rho\sin\theta\sin\phi)^2(\rho\cos(\phi))\cdot\rho^2\sin\phi\, d\phi\, d\theta\, d\rho$$

$$=\int_{\rho}\int_{\theta}\int_{\phi} \rho^5\sin^2\theta\sin^3\phi\cos\phi\, d\phi\, d\theta\, d\rho$$

Inner: $\int_{\phi=0}^{\pi/3} \rho^5\sin^2\theta\sin^3\cos\phi\, d\phi = \rho^5\sin^2\theta\int_{\phi=0}^{\pi/3}\sin^3\phi\cos\phi\, d\phi$

$$=\rho^5\sin^2\theta\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/3} = \rho^5\sin^2\theta\left(\frac{9}{64}\right)$$

Middle: $\int_{\theta=0}^{2\pi}\rho^5\sin^2\theta\,\frac{9}{64} = \frac{9}{64}\rho^5\int_{\theta=0}^{2\pi}\sin^2\theta\, d\theta$

$$=\frac{9}{128}\rho^5\int_{\theta=0}^{2\pi}1-\cos(2\theta)\, d\theta = \frac{9}{128}\rho^5\left[\theta-\frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

$$=\frac{18\pi}{128}\rho^5$$

Outer: $\int_{\rho=0}^{2}\frac{18\pi}{128}\rho^5 = \frac{18\pi}{128}\left[\frac{\rho^6}{6}\right]_0^2 = \frac{18\pi}{128}\left(\frac{64}{6}\right)$

$$\boxed{=\frac{3\pi}{2}}$$

Compute $\iiint_R 6xy\, dV$ $R = \{(x,y,z) : 0 \le y \le 1,\ y \le x \le 2y$
$0 \le z \le x+y\}$

$$\int_{y=0}^{1}\int_{x=y}^{2y}\int_{z=0}^{x+y} 6xy\ dz\,dx\,dy$$

Inner: $\int_{z=0}^{x+y} 6xy\, dz = 6xyz\Big|_{z=0}^{x+y} = 6xy(x+y) - 0$

$= 6x^2y + 6xy^2 = 6(x^2y + xy^2)$

Middle: $6\int_{x=y}^{2y} x^2y + xy^2\, dx = 6\left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_{x=y}^{2y}$

$= 6\left[\left(\frac{8y^4}{3} + \frac{4y^4}{2}\right) - \left(\frac{y^4}{3} + \frac{y^4}{2}\right)\right]$

$= 6y^4\left(\frac{8}{3} + \frac{4}{2} - \frac{1}{3} - \frac{1}{2}\right) = 6y^4\left(\underset{\frac{14}{6}}{\frac{7}{3}} + \underset{\frac{9}{6}}{\frac{3}{2}}\right) = \cancel{6}y^4\,\frac{23}{\cancel{6}}$

$= 23y^4$

Outer: $\int_{y=0}^{1} 23y^4 = 23\left[\frac{y^5}{5}\right]_{y=0}^{1} = \boxed{\frac{23}{5}}$

Ex: Compute $\int_{x=0}^{3} \int_{y=0}^{x} \int_{z=x-y}^{x+y} y \, dV$

Inner: $\int_{x-y}^{x+y} y \, dz = yz \Big|_{x+y}^{x+y} = y(x+y) - y(x-y)$

$= \cancel{yx} + y^2 - \cancel{xy} + y^2$

$= 2y^2$

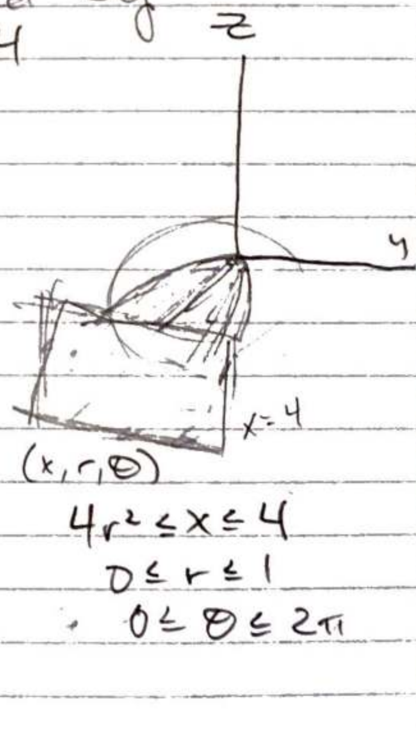
Middle: $\int_{y=0}^{x} 2y^2 \, dy = 2\frac{y^3}{3}\Big|_{y=0}^{x} = \frac{2}{3}x^3$

Inner: $\int_{x=0}^{3} \frac{2}{3}x^3 = \frac{2}{3}\frac{x^4}{4}\Big|_0^3 = \frac{2}{3}\frac{81}{4} = \frac{81}{6} = \boxed{\frac{27}{2}}$

Ex: $\int_R xy^2z \, dV$ where R is region bounded by $x = 4y^2 + 4z^2$ and $x = 4$

$\Rightarrow \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{x=4r^2}^{4} xy^2z \, dx \, dr \, d\theta$

$= \int_\theta \int_r \int_{x=4r^2}^{4} x(r\cos\theta)^2 r\sin\theta \, r \, dx \, dr \, d\theta$



(x, r, θ)

$4r^2 \leq x \leq 4$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

Inner: $\int_{x=4r^2}^{4} x r^4 \cos^2\theta \sin\theta \, dx$

$= r^4\cos^2\theta\sin\theta \left[\frac{x^2}{2}\Big|_{x=4r^2}^{4} \right]$

$= r^4\cos^2\theta\sin\theta \left[8 - 8r^4 \right]$

$= 8r^4\cos^2\theta\sin\theta - 8r^8\cos^2\theta\sin\theta$

Middle:

$8\cos^2\theta\sin\theta \int_{r=0}^{1} r^4 - r^8 \, dr = 8\cos^2\theta\sin\theta \left[\frac{r^5}{5} - \frac{r^9}{9} \right]_0^1$

$= \left(\frac{1}{5} - \frac{1}{9}\right) 8\cos^2\theta\sin\theta$

Outer: $\left(\frac{1}{5}-\frac{1}{9}\right)8\int_0^{2\pi}\sin\theta\cos^2\theta\,d\theta = 0$?

Exercise: $\int_R z\,dV$ V is bounded by $x^2+y^2=9$ and $y=3x$ in the first octant

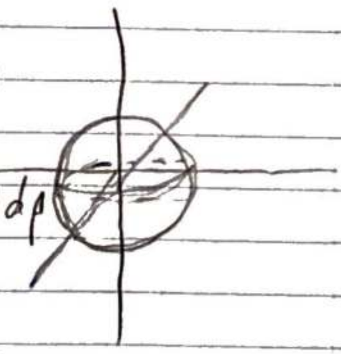
One Last Example in Spherical coordinates:

Ex: Compute the volume of the disk of radius $\alpha > 0$
→ We already did this in cartesian coordinates but it was difficult..

Sol: In spherical coord: $D_\alpha\{(\rho,\theta,\varphi)\}: 0\le\rho\le\alpha,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi$

$Vol(D_\alpha) = \iiint_{D^\alpha_{cart}} 1\,dV_{cart}$

$dV_{cart} = \rho^2\sin(\varphi)\,dV_{sp}$



$= \iiint_{D^\alpha_{sph}} 1\cdot\rho^2\sin(\varphi)\,dV_{sph} = \int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\rho^2\sin(\varphi)\,d\varphi\,d\theta\,d\rho$

$= \int_\varphi\int_\theta -\rho^2\cos(\varphi)\Big|_{\varphi=0}^{\pi}\,d\theta\,d\rho = \int_\rho\int_\theta -\rho^2(-1-1)\,d\theta\,d\rho$

$= \int_\rho\int_\theta 2\rho^2\,d\theta\,d\rho = 2\int_\rho \rho^2\left[\theta\right]_{\theta=0}^{2\pi}d\rho = 2\int_{\rho=0}^{\alpha}\rho^2(2\pi-0)\,d\rho$

$= 4\pi\left[\frac{1}{3}\rho^3\right]_{\rho=0}^{\alpha} = \frac{4}{3}\pi(\alpha^3-0) = \boxed{\frac{4}{3}\pi\alpha^3}$